Entanglement Entropy and Causal Set Theory

纠缠熵与因果集理论

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Contents

目录

Abstract

摘要

We review a formulation of the entanglement entropy of a quantum scalar field in terms of its spacetime two-point correlation functions. We discuss applications of this formulation to studying entanglement entropy in various settings in causal set theory. These settings include sprinklings of causal diamonds in various dimensions in flat spacetime, de Sitter spacetime, massless and massive theories, multiple disjoint regions, and nonlocal quantum field theories.

我们基于时空两点关联函数回顾了量子标量场纠缠熵的一种表述形式。我们讨论了该表述在因果集合论多个场景下研究纠缠熵的应用。这些场景包括: 平直时空不同维度下因果菱形的撒播、德西特时空、无质量与有质量理论、多个不相交区域, 以及非局域量子场论。

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Keywords

关键词

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纠缠熵 - 代数量子场论 - 协变正则化 - 关联函数 - 黑洞热力学 - 格林函数 · 时空域 · 非定域性 · 泊松涨落

Introduction

引言

Entanglement entropy is a useful measure of our limited access to quantum fields' degrees of freedom. This limited access can occur, for example, in the presence of an event horizon, where correlations between field values at points in the interior and exterior of the event horizon, $\langle 0|\phi(x_{int})\phi(x_{\text{ext}})|0\rangle$, are no longer available.

纠缠熵是衡量我们无法完全获取量子场自由度的有效度量。这种受限获取可出现在事件视界存在的场景中，例如，事件视界内外区域的场关联 $\langle 0|\phi(x_{int})\phi(x_{\text{ext}})|0\rangle$ 不再可被获取。

Entanglement entropy is one of the most important concepts in quantum gravity. It naturally combines both quantum (entanglement) and gravitational (area laws) properties. It was originally inspired by the search for a fundamental understanding of black hole entropy: we know that black holes classically have the Bekenstein-Hawking entropy [1, 2] associated with them, which scales as the spatial area of the event horizon, but we do not know what the fundamental or microscopic origin of this entropy is (e.g., in the statistical mechanical sense of what the microstates leading to this entropy are). It is one of the important tasks of quantum gravity to provide insight into this. Entanglement entropy, as first shown in [3], also generically scales as the area of the boundary of the entangling region (which in a black hole spacetime is the area of the event horizon). Hence, it is a promising direction to investigate this question. Ultimately, we expect entanglement entropy to contribute to black hole entropy; the open question is whether or not it will be the dominant contribution.

纠缠熵是量子引力中最重要的概念之一，它自然结合了量子 (纠缠) 和引力 (面积定律) 特性。它最初源于对黑洞熵基本本质的探索: 我们知道经典黑洞具有与之关联的贝肯斯坦-霍金熵 [1, 2] ，该熵随事件视界的空间面积标度变化，但我们尚不清楚这个熵的基本或微观起源 (例如，从统计力学角度看，产生该熵的微观态是什么)。量子引力的重要任务之一就是为这个问题提供解答。正如文献 [3] 首次指出的，纠缠熵也通常随纠缠区域边界的面积标度 (在黑洞时空中等价于事件视界的面积)。因此，研究这个问题是一个很有前景的方向。我们最终期望纠缠熵可以解释黑洞熵，悬而未决的问题是它是否会是黑洞熵的主要贡献来源。

Let us now review some general aspects of entanglement entropy. Entanglement entropy is conventionally defined using a density matrix ρ , which is initially pure, meaning that we have full information about the quantum system and the von Neumann entropy vanishes:

下面我们来回顾纠缠熵的一些基本性质。按惯例，纠缠熵通过密度矩阵 ρ 定义，初始系统是纯态，即我们掌握该量子系统的全部信息，此时冯·诺依曼熵为零:

$$S=-\operatorname{Tr}\rho_{\text{pure}}\ln\rho_{\text{pure}}=0. \tag{1}$$

Subsequently, we trace out of the density matrix the parts of the system that we do not have information about. Traditionally this tracing out is done on a spatial hypersurface $\sum$, as in Fig. 1, which is divided into two complementary subregions, region A and region B , one of which represents the degrees of freedom we do have access to and the other the degrees of freedom that we do not have access to. After we trace out the degrees of freedom in one of the subregions, for example, those in B (We would get the same answer if we instead traced out the degrees of freedom in A to get ρ_B and computed S_B . We refer to this as complementarity below.), we get a reduced density matrix:

随后，我们对密度矩阵中我们无法获取信息的系统部分做求迹操作。传统上这个求迹是在空间超曲面上完成的 $\sum$ ，如图 1 所示，该超曲面被分为两个互补子区域: 区域 A 和区域 B ，其中一个对应我们可获取的自由度，另一个对应我们无法获取的自由度。我们对其中一个子区域的自由度做求迹后，例如对 B 的自由度做求迹 (如果我们改为对 A 的自由度做求迹得到 ρ_B 再计算 S_B ，会得到相同的结果，下文中我们称之为互补性)，我们得到约化密度矩阵:

$$\rho_A=\operatorname{Tr}_B\rho, \tag{2}$$

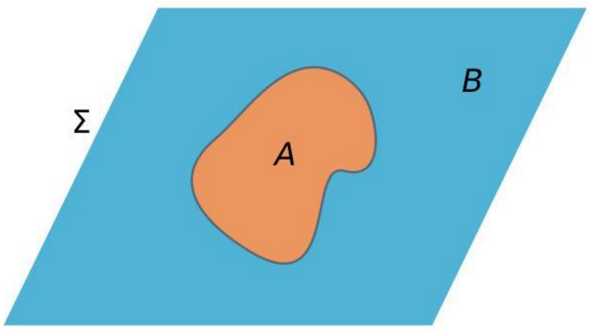
and the entropy of entanglement between region A and B is defined as [3]

区域 A 和区域 B 之间的纠缠熵定义为 [3]

$$S_A=-\operatorname{Tr}\rho_A\ln\rho_A. \tag{3}$$

Fig. 1 A spatial hypersurface $\sum$ divided into two complementary subregions A and B

图 1 空间超曲面 $\sum$ 被分为两个互补子区域 A 和 B



It is crucial that the theory one is working with has a UV cutoff with respect to which we count how many degrees of freedom we do or do not have access to. Without a UV cutoff, we would get an infinite answer for the entanglement entropy in (3).

> 关键在于，我们研究的理论需要存在紫外截断，我们依靠这个截断来统计可获取和不可获取的自由度数量。如果没有紫外截断，我们得到的 (3) 式中纠缠熵会是无穷大。

Since the original work [3] in the context of black hole entropy, entanglement entropy has found many additional useful applications in other areas of physics such as quantum information [4] and condensed matter physics [5]. Depending on the specific application in mind, different techniques may be used to evaluate the entropy in (3). This choice of technique is often motivated by physical and computational considerations.

> 自文献 [3] 在黑洞熵背景下的开创性工作以来，纠缠熵已经在物理的其他领域得到了诸多有用应用，例如量子信息 [4] 和凝聚态物理 [5]。根据具体应用场景的不同，人们可以使用不同的技术计算 (3) 式中的熵。技术的选择通常由物理和计算层面的考量决定。

In causal sets, there is no analogue of field data on a spatial hypersurface (i.e., a Cauchy surface). Figure 2 illustrates the reason for this, which is essentially that there is no guarantee that there will not be relations that will not make an imprint on a subset of unrelated elements. Therefore, we cannot work on a hypersuface as in Fig. 1 and must use an intrinsically spacetime approach to compute the entanglement entropy. Fortunately, a spacetime definition of entanglement entropy, in terms of the spacetime two-point correlation function, exists and can be used in causal set calculations. We review this formulation in section ”Entanglement Entropy from Spacetime Two-Point Correlation Functions”. While we are led to a spacetime formulation of entanglement entropy in causal set theory out of necessity, there are in fact numerous attractive features of working with a spacetime definition of entanglement entropy. For example, quantum fields themselves are really spacetime quantities (their domain is spacetime). Therefore, it is more natural to study them in spacetime. Furthermore, quantum fields are highly singular and may not always admit meaningful restrictions to spatial hypersurfaces. We devote section ”Quantum Field Operators Are Distributions in Spacetime” to supporting these statements with some concrete examples. Finally, in section ”Applications” we discuss several calculations of entanglement entropy, in settings including sprinklings into flat spacetime and de Sitter spacetime, before ending in section ”Discussion and Outlook” with a discussion of some subtleties of and the future directions for this work.

> 在因果集中，空间超曲面 (即柯西曲面) 上不存在类似场数据的对应物。图 2 说明了原因，本质在于无法保证不存在不会在无关元素子集上留下印记的关系。因此我们无法像图 1 那样在超曲面上开展工作，必须采用本质上属于时空的方法来计算纠缠熵。幸运的是，基于时空两点关联函数的纠缠熵时空定义已经存在，可用于因果集计算。我们在“时空两点关联函数导出的纠缠熵”一节回顾了这一表述。虽然我们是出于必要性才在因果集理论中得到了纠缠熵的时空表述，但采用纠缠熵的时空定义实际上具备诸多吸引人的优势。例如，量子场本身本质上就是时空量 (其定义域就是时空)，因此在时空中研究它们更加自然。此外，量子场具有高度奇异性，并非总能对空间超曲面给出有意义的限制。我们在“量子场算符是时空中的分布”一节中通过具体例子佐证了这些观点。最后，在“应用”一节中，我们讨论了在平直时空和德西特时空撒播等多个场景下的若干纠缠熵计算，随后在“讨论与展望”一节中讨论了这项工作存在的一些微妙问题，并指明了未来研究方向。

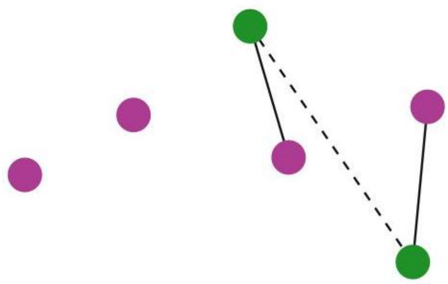


Fig. 2 The pink elements form a maximal antichain, which is an inextendible set of unrelated elements such that every other element is to the past or future of at least one of the elements of this set. This is the analogue of a spatial hypersurface in a causal set. In order for it to be a viable analogue of a Cauchy surface, we must be able to deduce all causal relations using relations involving the antichain elements. However, this is not possible, as illustrated by the counterexample above: the causal relation represented by the dashed line has no imprint on the antichain; hence, we do not have an analogue of a Cauchy hypersurface in a causal set. The causal interval between the two green elements does not include any element of the antichain, and the dashed line between the pair of green elements represents a link. The figure above is to be understood as a subset of a causal set, with all elements not shown causally related to the pink elements in this diagram, by the definition of a maximal antichain

图 2 粉色元素构成了一个极大反链，即一个不可扩张的无关元素集合，满足所有其他元素都位于该集合中至少一个元素的过去或未来。这就是因果集中空间超曲面的对应物。要让它成为柯西曲面的有效对应，我们必须能够通过涉及反链元素的关系推导出所有因果关系。但正如上方反例所示，这是无法实现的: 虚线表示的因果关系没有在反链上留下任何印记；因此，因果集中不存在柯西超曲面的对应物。两个绿色元素之间的因果区间不包含任何反链元素，两个绿色元素之间的虚线代表一条关联。根据极大反链的定义，上图应理解为因果集的一个子集，所有未展示的元素都与图中的粉色元素存在因果关联。

Entanglement Entropy from Spacetime Two-Point Correlation Functions

来自时空两点关联函数的纠缠熵

A quantum field theory with local operators is typically fully determined by the set of all its n-point correlation functions. If we consider a scalar field ϕ , these are $\langle 0\,|\phi\,(x_1)\,\phi\,(x_2)|\,0\rangle$, ... $\langle 0\,|\phi\,(x_1)\,\phi\,(x_2)\,...\,\phi\,(x_n)|\,0\rangle$, where $\{x_1, x_2, ...\}$ represent points in spacetime. In a Gaussian quantum field theory, things are much simpler because we only need to know the two-point correlation function $\langle 0\,|\phi\,(x_1)\,\phi\,(x_2)|\,0\rangle$. In this case, all the higher n-point correlation functions are derivable from the two-point function through Wick's theorem. For the remainder of this chapter, except where explicitly mentioned otherwise, we will restrict the discussion to Gaussian scalar field theories.

包含局域算符的量子场论通常完全由其所有 n 点关联函数的集合确定。如果考虑一个标量场 ϕ，这些关联函数为 $\langle 0|\phi(x_1)\phi(x_2)|0\rangle, \ldots \langle 0|\phi(x_1)\phi(x_2)\ldots\phi(x_n)|0\rangle$ ，其中 $\{x_1, x_2, \ldots\}$ 表示时空中的点。在高斯量子场论中，情况简单得多，因为我们只需要知道两点关联函数 $\langle 0|\phi(x_1)\phi(x_2)|0\rangle$ 。在这种情况下，所有更高阶的 n 点关联函数都可以通过威克定理从两点函数推导得出。本章剩余部分中，除非另有明确说明，我们都将讨论限定在高斯标量场论范围内。

Entropy from Correlation Functions

关联函数导出的熵

Since we are focusing on a Gaussian scalar field theory, as mentioned above, everything (including the entanglement entropy (3)) must be expressible in terms of the two-point correlation function $\langle 0|\phi(x)\phi(x')|0\rangle$. The definition of entropy introduced in [6], which we review in this subsection, does precisely that. Specifically, the entropy is given by the sum:

由于我们聚焦于高斯标量场论，正如前文所述，所有量 (包括纠缠熵 (3)) 都必须可以通过两点关联函数 $\langle 0|\phi(x)\phi(x')|0\rangle$ 表示。我们在本小节回顾的、文献 [6] 中提出的熵定义恰好满足这一点。具体而言，熵由下式给出:

$$S = \sum_{\lambda} \lambda \ln|\lambda| \tag{4}$$

over the solutions λ to the generalized eigenvalue problem

对广义特征值问题的解 λ 求和

$$Wv = i\lambda\Delta v \tag{5}$$

while excluding components in the kernel of Δ

同时排除 Δ 核中的分量

$$\Delta v \neq 0\,. \tag{6}$$

W in (5) is the spacetime two-point correlation function or Wightman function, $W(x,x') = \langle 0|\phi(x)\phi(x')|0\rangle$, and $i\Delta$ is the Pauli-Jordan function or spacetime commutator of the field, $i\Delta(x,x') = [\phi(x), \phi(x')]$. In a Gaussian theory, $i\Delta$ is a c-number. We can obtain Δ using the retarded Green function, G_R , as $\Delta(x,x') = G_R(x,x') - G_R(x',x)$. If we already have a state and therefore a W to work with, we can also obtain it through the imaginary or antisymmetric part of W , i.e., $i\Delta(x,x') = W(x,x') - W(x',x) = 2\,\mathrm{Im}(W(x,x'))$.

(5) 式中的 W 是时空两点关联函数即威格曼函数 $W(x,x') = \langle 0|\phi(x)\phi(x')|0\rangle$, $i\Delta$ 是泡利-约当函数，也就是场的时空对易子 $i\Delta(x,x') = [\phi(x), \phi(x')]$ 。在高斯理论中，$i\Delta$ 是一个 c 数。我们可以利用推迟格林函数 G_R 得到 Δ , 形式为 $\Delta(x,x') = G_R(x,x') - G_R(x',x)$ 。如果已经给定一个态以及对应的 W , 也可以通过 W 的虚部或反对称部分得到 $i\Delta$, 即 $i\Delta(x,x') = W(x,x') - W(x',x) = 2\,\mathrm{Im}(W(x,x'))$.

More precisely, we must start with a global state (and its corresponding W) which is initially pure. In terms of the eigenvalue equation (5) and entropy (4), purity would mean solutions λ that are 1’ s and 0’ s. In the next subsection, we review one choice of pure state, the Sorkin-Johnston state, which can be defined in causal set theory. With a pure state at hand, we subsequently exclude parts of the quantum system that we don’ t have access to by restricting the elements x and x' in $W(x,x')$ and $i\Delta(x,x')$ to lie in the spacetime subregion that we do have access to, before solving (5). Then, we can interpret the resulting entropy from substituting the solutions into (4) as the entanglement entropy between the spacetime region which is the domain of x and x' and its causal complement. Note that the causal complement will not in general be the complementary spacetime domain (see, e.g., Fig. 3).

> 更准确地说，我们必须从一个整体纯态 (以及其对应的 W) 出发。就特征值方程 (5) 和熵 (4) 而言，纯态意味着解 λ 只能取 1 或 0。在下一小节中，我们将回顾一类满足要求的纯态——索尔金-约翰斯顿态，它可以在因果集合论中定义。得到纯态后，我们在求解 (5) 之前，先把 $W(x,x')$ 和 $i\Delta(x,x')$ 中的元素 x 和 x' 限制在我们可观测的时空子区域内，以此去掉量子系统中我们无法触及的部分。之后，我们可以将解代入 (4) 得到的熵解释为: x 和 x' 所在的时空区域与其因果补区域之间的纠缠熵。注意，因果补一般不等于时空补区域 (参见例如图 3)。

It is in this way, owing to $W(x,x')$ and $i\Delta(x,x')$ being spacetime functions, that this formulation of entanglement entropy is a spacetime approach. Additionally, due to its spacetime nature, it allows one to use a spacetime UV cutoff such as the discreteness scale of a causal set. This can give our counting of degrees of freedom a covariance and universality that is not present in the more spatial formulations.

> 正是由于 $W(x,x')$ 和 $i\Delta(x,x')$ 都是时空函数，这种纠缠熵表述才是一种时空方法。此外，由于其时空属性，该表述允许我们使用时空紫外截断，例如因果集合的离散标度。这可以给我们的自由度计数带来协变性与普适性，这是空间表述通常不具备的性质。

We can in some cases compare the results obtained from this method to the results from the conventional spatial methods. We can do this, for example, when the regions of spacetime that we consider are domains of dependence of Cauchy surfaces. Figure 3 illustrates an example of this, since the causal diamonds are domains of dependence of the 1D intervals connecting the left and right spatial corners (e.g., their diameters).

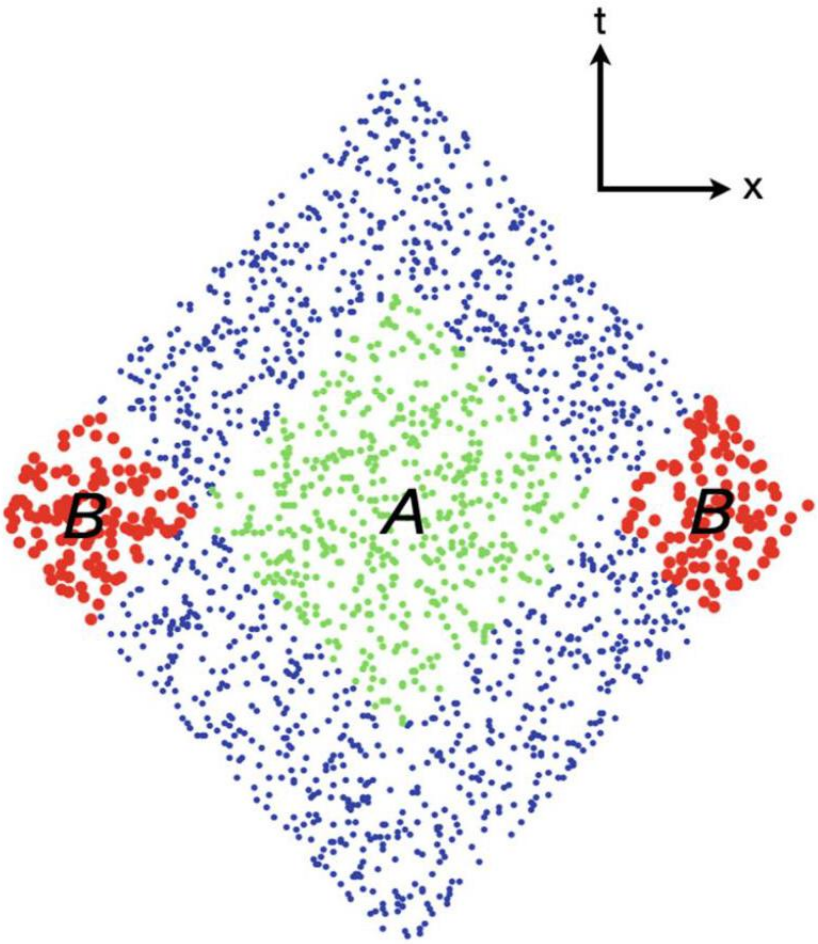
> 在某些情况下，我们可以将该方法得到的结果与传统空间方法得到的结果进行比较。例如，当我们考虑的时空区域是柯西曲面的依赖域时就可以进行这种比较。图 3 给出了一个相关示例，因为因果菱形就是连接左右空间角的一维区间 (例如直径) 的依赖域。

This entropy formulation is derived in [6]. An alternative derivation of it using the replica trick can be found in [7]. Quite surprisingly, this formula can actually be used beyond Gaussian theories. In [7] it was shown that up to first order in perturbation theory, the entanglement entropy of even non-Gaussian or interacting theories, is captured by the two-point correlation function via (5) and (4). The only difference in the interacting theory case is that the W that enters (5) is the interacting one, and while $i\Delta$ would still be the antisymmetric part of W , it is no longer the commutator.

该熵公式推导自文献 [6]。可在文献 [7] 中找到使用复制技巧对其进行的另一推导。相当令人惊讶的是，该公式实际上可应用于高斯理论之外的情况。文献 [7] 表明，直至微扰论一阶，即便是非高斯理论或相互作用理论，其纠缠熵也可通过式 (5) 和 (4) 由两点关联函数得到。相互作用理论情况的唯一区别是，代入式 (5) 的 W 是相互作用理论的 W，且尽管 $i\Delta$ 仍是 W 的反对称部分，但它不再是对易子。

Fig. 3 Causally complementary spacetime subregions A and B in a causal set

图 3 因果集中的因果互补时空子区域 A 和 B



The Sorkin-Johnston (SJ) Vacuum

索金-约翰斯顿 (SJ) 真空

The Sorkin-Johnston (SJ) Wightman function [8,9] is defined in the same algebraic and spacetime spirit as the entropy formulation we have just reviewed. It uniquely picks out a vacuum state in any globally hyperbolic spacetime.

索金-约翰斯顿 (SJ) 怀特曼函数 [8,9] 遵循我们刚刚回顾的熵表述的代数与时空思想定义。它能在任意整体双曲时空中唯一选出一个真空态。

To define the SJ state, we require the spacetime commutator function $i\Delta$. As we saw above, we can express $i\Delta$ in terms of the retarded Green function $\left(\left(\Box + m^2\right) \; G_R\left(x, x'\right) = -\frac{\delta^D(x-x')}{\sqrt{-g}}\right.$, where $G_R\left(x, x'\right)$ is only nonzero if $x' \prec x$.) as

要定义 SJ 态，我们需要时空对易子函数 $i\Delta$ 。正如我们上文所见，我们可以用推迟格林函数 $\left(\left(\Box + m^2\right) \; G_R\left(x, x'\right) = -\frac{\delta^D(x-x')}{\sqrt{-g}}\right.$ 表示 $i\Delta$ ，其中仅当 $x' \prec x$ 时 $G_R\left(x, x'\right)$ 非零。) 写作

$$\Delta\left(x, x'\right) = G_R\left(x, x'\right) - G_R\left(x', x\right). \tag{7}$$

$i\Delta$ is antisymmetric and Hermitian. We can then rewrite it as an expansion in terms of its positive and negative eigenvalues and their respective eigenfunctions as

$i\Delta$ 是反对称且厄米的。我们可以随后将其改写为关于正负特征值及对应本征函数的展开式，形式为

$$i\Delta = \sum_i \tilde{\lambda}_i u_i u_i^\dagger - \tilde{\lambda}_i v_i v_i^\dagger, \tag{8}$$

where u_i and v_i are the normalized positive and negative eigenvectors, respectively, and $\widetilde{\lambda_i} > 0$. The SJ Wightman function is defined by restricting to the positive eigenspace of $i\Delta$:

其中 u_i 和 v_i 分别是归一化的正、负本征向量，且 $\widetilde{\lambda_i} > 0$ 。SJ 怀特曼函数通过限制在 $i\Delta$ 的正特征空间上定义:

$$W_{SJ} \;:=\; \mathrm{Pos}\left(i\Delta\right) = \sum_i \tilde{\lambda}_i u_i u_i^\dagger. \tag{9}$$

When we define W in this way, the only solutions to the generalized eigenvalue equation (5) are $\lambda = 0$ or $\lambda = 1$, and this makes the entropy (4) vanish, as required if the state is pure. It is also worth mentioning that in static spacetimes, the SJ vacuum is the same one that is picked out by the timelike and hypersurface-orthogonal Killing vector [10]; hence, the SJ state is known to reflect the symmetries of the background geometry if there are any. In all of the entanglement entropy applications that we will review in section ”Applications”, the SJ state is used as the pure state.

当我们以这种方式定义 W 时，广义特征方程 (5) 的唯一解是 $\lambda = 0$ 或 $\lambda = 1$ ，这使得熵 (4) 为零，符合纯态的要求。值得一提的是，在静态时空中，SJ 真空与类时且正交于超曲面的基灵矢量选出的真空一致 [10]；因此，若背景几何存在对称性，已知 SJ 态会反映这些对称性。我们将在“应用”一节回顾的所有纠缠熵应用中，都采用 SJ 态作为纯态。

Both the entanglement entropy formulation and SJ state prescription can also be used in continuum spacetimes. Before moving on to their applications in causal set theory, we will in the next section motivate why it is desirable to work with spacetime formulations, in general, when studying quantum field theories.

纠缠熵表述与 SJ 态方案都可用于连续时空。在讨论它们在因果集理论中的应用之前，我们会在下一节说明，在研究量子场论时，总体而言使用时空表述为什么更合理。

Quantum Field Operators Are Distributions in Spacetime

量子场算符是时空中的分布

At least as early as a 1933 work by Bohr and Rosenfeld [11], it has been recognized that quantum field operators are only well-defined as averages over finite regions of spacetime rather than at individual spacetime points. This averaging is achieved through smearing the quantum field with smooth, real-valued functions with compact support:

早在玻尔与罗森菲尔德 1933 年的工作 [11] 中，人们就已经认识到: 量子场算符仅作为有限时空区域而非单个时空点上的平均是良定义的，这种平均通过将量子场与紧支撑的实值光滑函数做抹除得到:

$$\phi(f) = \int \phi(x) f(x)\, dV. \tag{10}$$

See also [12] for a modern review of the convergence issues that arise if quantum fields are defined at points in spacetime rather than as distributions. Less wellknown is the fact that generic nonlinear operators in free quantum field theories are not well-defined if they are smeared with functions with support only on spatial hypersurfaces rather than spacetime regions [13, 14]. They become well-defined only after some smearing in a duration of time as well. Furthermore, based on perturbation theory, then, it is expected that operators in generic interacting quantum field theories are also not well-defined on spatial hypersurfaces, since they would contain the ill-defined nonlinear terms from the free theory.

若将量子场定义在时空点上而非作为分布，会产生收敛性问题，相关现代综述参见 [12]。鲜为人知的一点是: 在自由量子场论中，若一般非线性算符仅被支撑在空间超曲面而非时空区域上的函数抹除，那么它不是良定义的 [13, 14]，这类算符只有额外在一段时间上做抹除后才会成为良定义。进一步，基于微扰论可以推知: 一般相互作用量子场论中的算符也无法在空间超曲面上良定义，因为这类算符会包含自由理论中本身定义不良的非线性项。

Below we study this behavior for two generic nonlinear operators (ϕ^2 and T_{ab}) in free scalar field theory in Minkowski spacetime.

下文我们将研究闵氏时空自由标量场论中，两个一般非线性算符 (ϕ^2 和 T_{ab}) 的该性质。

Variance of a Free Scalar Field ϕ^2 Operator

自由标量场 ϕ^2 算符的方差

Consider the usual quantum scalar field operator in $3+1$ spacetime dimensions:

考虑 3+1 维时空下的常规量子标量场算符:

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{ip\cdot x} + a_{\mathbf{p}}^\dagger e^{-ip\cdot x} \right), \tag{11}$$

where $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$ and $x = (t, \mathbf{x})$.

其中 $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$ 和 $x = (t, \mathbf{x})$。

Let us now examine the normal-ordered ϕ^2 operator, and smear it with a test function $f(x) = f_1(t) f_2(\mathbf{x})$ with compact support on a time $t' =$ const slice, i.e., $f_1(t) = \delta(t - t')$

现在我们来研究正规序下的 ϕ^2 算符，用测试函数 $f(x) = f_1(t) f_2(\mathbf{x})$ 对其做抹除，该测试函数在时刻 $t' =$ 为常数的类空切片上具有紧支撑，即 $f_1(t) = \delta(t - t')$

$$: \phi^2(t', f) \; := \int dt d^3 x f_2(\mathbf{x}) \delta(t - t') \int \frac{d^3 p d^3 k}{2(2\pi)^6 \sqrt{E_{\mathbf{p}} E_{\mathbf{k}}}} \left(a_{\mathbf{p}} a_{\mathbf{k}} e^{i(p+k)\cdot x} + \ldots \right).$$

(12)

We then square the result and compute its expectation value in the Minkowski vacuum:

我们随后对结果取平方，计算它在闵可夫斯基真空中的期望值:

$$\langle 0 | : \phi^2(t', f_2) :: \phi^2(t', f_2) : | 0 \rangle = \int \frac{d^3 p d^3 k \left| \widetilde{f}_2(\mathbf{k} + \mathbf{p}) \right|^2}{2(2\pi)^{12} E_{\mathbf{p}} E_{\mathbf{k}}}, \tag{13}$$

where $\widetilde{f}_2$ is the Fourier transform of f_2 . The integral can be seen to be linearly divergent by simply counting the powers of $\mathbf{p}$ and $\mathbf{k}$, and assuming that f_2 is square-integrable. However, (13) converges if the smearing is done over an extent in time as well. To see this, let us for simplicity assume that the smearing function is separable, i.e., $f(x) = f_1(t) f_2(\mathbf{x})$ as before (where f_1 is no longer a delta function) and both f_1 and f_2 are square-integrable. Then,

其中 $\widetilde{f}_2$ 是 f_2 的傅里叶变换。仅通过数 $\mathbf{p}$ 和 $\mathbf{k}$ 的幂次，同时假设 f_2 是平方可积的，就能看出该积分是线性发散的。不过，如果抹除也在时间方向上延伸一定范围，(13) 式就会收敛。为了说明这一点，我们为简化计算假设抹除函数是可分离的，即和之前一样为 $f(x) = f_1(t) f_2(\mathbf{x})$ (其中 f_1 不再是 delta 函数)，且 f_1 和 f_2 都是平方可积的。由此可得:

$$\langle 0 | : \phi^2(f) :: \phi^2(f) : | 0 \rangle \propto \int \frac{d^3 p d^3 k \left| \widetilde{f}_2(\mathbf{p} + \mathbf{k}) \right|^2 \left| \widetilde{f}_1(p^0 + k^0) \right|^2}{p^0 k^0} \tag{14}$$

$$\leq \int \frac{d^3 k I(\mathbf{k}) \widetilde{F}_1(k^0)^2}{\sqrt{|\mathbf{k}|^2 + m^2}}, \tag{15}$$

with $\widetilde{F}_1\left(k^0\right)$ defined as the absolute maximum of $\widetilde{f}_1$ with respect to p^0 at a given k^0 , and $I\left(\mathbf{k}\right)$ being the p -integral of the remaining p -dependent variables. Since it can be shown that $I\left(\mathbf{k}\right) \sim \frac{1}{|\mathbf{k}|}$ for large $|\mathbf{k}|$, and $\widetilde{F}_1$ is square-integrable, we see by counting powers of k that the integral is convergent.

其中 $\widetilde{F}_1\left(k^0\right)$ 定义为固定 k^0 时，$\widetilde{f}_1$ 关于 p^0 的绝对最大值，$I\left(\mathbf{k}\right)$ 是剩余依赖 p 的变量对 p 的积分。由于可以证明对于大的 $|\mathbf{k}|$ 有 $I\left(\mathbf{k}\right) \sim \frac{1}{|\mathbf{k}|}$ ，且 $\widetilde{F}_1$ 是平方可积的，我们通过数 k 的幂次可知该积分是收敛的。

Expectation values such as the one we have considered above are ubiquitous in operator product expansions (OPEs) and appear in generic correlation functions. Similar divergences occur for other nonlinear operators such as ϕ^n for $n \geq 2$ and the stress energy tensor T_{ab} (as we show below), and suggest that quantum field theories generally need to be considered in a spacetime region.

我们上述讨论的这类期望值在算符乘积展开 (OPE) 中十分常见，也出现在一般关联函数中。对于其他非线性算符也会出现类似的发散，例如 $n \geq 2$ 对应的 ϕ^n ，还有应力能量张量 T_{ab} (我们下文会证明)，这说明量子场论通常都需要在特定时空区域内讨论。

Variance of Scalar Field Energy Density Too

标量场能量密度的方差也如此

The simple construction above shows that certain operator expectation values are divergent when smeared only on a spatial hypersurface and convergent when smeared in spacetime. Let us next consider the analogous calculation for a physically more interesting quantity, namely, the scalar field's stress-energy tensor:

上述简单构造表明，部分算符期望值仅在空间超曲面上 smear 时发散，在时空中 smear 后收敛。下面我们对一个物理上更具意义的量——标量场的能量动量张量做类似计算:

$$: T_{ab} \; := \; : \partial_a\phi\partial_b\phi \; : -\frac{1}{2}\eta_{ab} \; : \partial_c\phi\partial^c\phi \; : -\frac{m^2}{2}\eta_{ab} \; : \phi^2 \; : \tag{16}$$

We will focus our attention on the energy density component $: T_{00}$:, and show that its variance diverges when smeared on a hypersurface, but can be made convergent with an appropriate spacetime smearing. Substituting in the definition of ϕ and smearing with a square-integrable test function $f\left(x\right) = \delta\left(t-t'\right) f_2\left(\mathbf{x}\right)$ defined on a timeslice, the expectation value of the variance is given by

我们将重点关注能量密度分量: T_{00} :，并证明它在超曲面上 smear 后方差发散，而通过合适的时空 smear 可以使其收敛。代入 ϕ 的定义，并用定义在时间切片上的平方可积测试函数 $f\left(x\right) = \delta\left(t-t'\right) f_2\left(\mathbf{x}\right)$ 做 smear 后，方差的期望值可写为

$$\left\langle 0 \right| : T_{00}\left(f\right) :{}^2 \left| 0 \right\rangle = \int \frac{d^3p d^3k}{8(2\pi)^{12}} \left(\frac{E_{\mathbf{p}} E_{\mathbf{k}} + \mathbf{p}\cdot\mathbf{k} - m^2}{\sqrt{E_{\mathbf{p}}}\sqrt{E_{\mathbf{k}}}} \right)^2 \left| \widetilde{f}_2\left(\mathbf{p}+\mathbf{k}\right) \right|^2 . \tag{17}$$

We then see in the large $|\mathbf{p}|$ and $|\mathbf{k}|$ limit that the leading order contribution to the integral is

我们随后会看到，在大 $|\mathbf{p}|$ 和 $|\mathbf{k}|$ 极限下，积分的领头阶贡献为

$$\langle 0|: T_{00}(f):^2|0\rangle \sim \int d^3p d^3k\, |\mathbf{p}|\, |\mathbf{k}|\, (1+\cos\theta)^2 \left|\widetilde{f_2}(\mathbf{p}+\mathbf{k})\right|^2, \tag{18}$$

θ being the angle between $\mathbf{p}$ and $\mathbf{k}$ in the dot product above. Since this integral has higher powers of $\mathbf{p}$ and $\mathbf{k}$ than the integral in (13) which we already showed diverges, it follows trivially that (18) diverges as well.

θ 是上述点积中 $\mathbf{p}$ 和 $\mathbf{k}$ 的夹角。由于该积分比我们已经证明发散的式 (13) 中的积分包含更高次幂的 $\mathbf{p}$ 和 $\mathbf{k}$，显然可以推出式 (18) 也发散。

In order for a time smearing to make (18) converge, more care needs to be taken than in (14) where generic square-integrable smearing functions suffice. Here we have in effect four extra factors of $\mathbf{p}$ and $\mathbf{k}$, so we need to ensure that the Fourier transforms of our smearing functions decay fast enough near infinity to counteract this. For this reason and for simplicity of analysis, we will use Gaussian smearing functions (While technically a Gaussian does not have compact support, its rapid decay restricts meaningful values to a local enough region. In any case, one could also work with ”bump” functions which have compact support, but nonetheless have Fourier transforms that decay faster than any power law, as demonstrated in [15].). We again write the spacetime smearing function as the separable function $f(t,\mathbf{x}) = f_1(t) f_2(\mathbf{x})$, where

与只需要通用平方可积 smear 函数就足够的式 (14) 相比，要让时间 smear 让式 (18) 收敛，我们需要更谨慎的处理。实际上此处我们多了四个 $\mathbf{p}$ 和 $\mathbf{k}$ 因子，因此我们需要保证 smear 函数的傅里叶变换在无穷附近衰减足够快，来抵消发散。出于这个原因，同时为简化分析，我们将使用高斯 smear 函数 (尽管严格来说高斯函数不具有紧支撑，但其快速衰减将有效有意义的值限制在足够局域的区域内。无论如何，我们也可以使用具有紧支撑的“凸包”函数，正如文献 [15] 所证明的，这类函数的傅里叶变换衰减速度快于任意幂律)。我们再次将时空 smear 函数写成分离函数 $f(t,\mathbf{x}) = f_1(t) f_2(\mathbf{x})$ ，其中

$$f_1(t) = e^{-at^2} \Rightarrow \left|\widetilde{f_1}(p^0+k^0)\right|^2 \propto e^{-\frac{(p^0+k^0)^2}{2a}}, \tag{19}$$

$$f_2(\mathbf{x}) = e^{-b\mathbf{x}^2} \Rightarrow \left|\widetilde{f_2}(\mathbf{p}+\mathbf{k})\right|^2 \propto e^{-\frac{(\mathbf{p}+\mathbf{k})^2}{2b}}. \tag{20}$$

The variance of the spacetime smeared energy density then at large $|\mathbf{p}|$ and $|\mathbf{k}|$ has leading order contribution:

经过时空 smear 的能量密度的方差，在大 $|\mathbf{p}|$ 和 $|\mathbf{k}|$ 下的领头阶贡献为:

$\langle 0|$

$$\langle 0|: T_{00}(f):^2|0\rangle \sim \int d^3p d^3k\, |\mathbf{p}|\, |\mathbf{k}|\, (1+\cos\theta)^2 \left|\widetilde{f_1}(p^0+k^0)\right|^2 \left|\widetilde{f_2}(\mathbf{p}+\mathbf{k})\right|^2$$

(21)

$$\lesssim \int d^3k\, |\mathbf{k}|\, \widetilde{f_1}(k^0)^2 I(\mathbf{k}), \tag{22}$$

where the inequality follows from $\widetilde{f_1}(v+w) < \widetilde{f_1}(v)$ for all positive v, w when $\widetilde{f_1}$ is a Gaussian, and we have simply grouped the remaining terms into the p integral I. One can then asymptotically evaluate I as

当 $\widetilde{f_1}$ 为高斯函数时，对所有正的 v, w 都满足 $\widetilde{f_1}(v+w) < \widetilde{f_1}(v)$ 因此得到上述不等式，我们仅将剩余项整理为 p 积分 I。随后可以对 I 做渐近求值得到

$$I(\mathbf{k}) \propto \int d^3p\, |\mathbf{p}| \left(1+\cos\theta_p\right)^2 e^{-\frac{(\mathbf{p}+\mathbf{k})^2}{2b}} \tag{23}$$

$$\sim \mathcal{O}\left(|\mathbf{k}|^4 \log(|\mathbf{k}|)\right) + \mathcal{O}(|\mathbf{k}|) + \mathcal{O}\left(|\mathbf{k}|^2 e^{-\frac{\mathbf{k}^2}{2b}}\right), \tag{24}$$

in the large $|\mathbf{k}|$ limit. Replacing the leading-order term back into (22), the upper bound on the variance of the energy density becomes asymptotic to the $|\mathbf{k}|$ -integral:

在大 $|\mathbf{k}|$ 极限下。将领头阶项代回式 (22)，能量密度方差的上界渐近等于 $|\mathbf{k}|$ 积分:

$$\langle 0|:T_{00}(f):^2|0\rangle \lesssim \int_0^\infty d|\mathbf{k}|\, |\mathbf{k}|^7 \log(|\mathbf{k}|)\, e^{-\frac{|\mathbf{k}|^2}{2a}}. \tag{25}$$

The decaying Gaussian will dominate asymptotically, so this integral is finite.

衰减的高斯函数会在渐近处占主导，因此该积分是有限的。

Therefore, as anticipated, $\langle 0|:T_{ab}::T_{ab}:|0\rangle$ is well-defined only as a distribution in spacetime. The smearing in the above discussion can be considered as a model for making a measurement. We would certainly not expect the energy in a bounded region to have an infinite variance, but we see that a finite variance is only obtained if the bounded region is in spacetime rather than in space alone. Hence, we can conclude that the quantum stress tensor is only meaningful as a distribution in spacetime. See also [16] for evidence that a well-defined probability distribution for the quantum stress tensor is only obtained for averages in time or spacetime.

因此，正如预期，$\langle 0|:T_{ab}::T_{ab}:|0\rangle$ 仅作为时空中的分布是良定义的。上述讨论中的 smear 可以被视为测量的模型。我们当然不会预期有界区域内的能量具有无穷大方差，但我们看到仅当有界区域是时空区域而非纯空间区域时，我们才能得到有限方差。因此我们可以得出结论: 量子能量动量张量仅作为时空中的分布是有意义的。关于量子能量动量张量仅在时间或时空平均下才能得到良定义概率分布的证据，另见文献 [16]。

Applications

应用

Having established the importance of entanglement entropy in quantum gravity, as well as the importance of treating quantum fields in a spacetime domain, let us now explore what work on entanglement entropy in causal set theory has been done.

在我们已经明确了纠缠熵在量子引力中的重要性，以及在时空域中处理量子场的重要性之后，现在让我们探索因果集理论中关于纠缠熵已经完成的相关工作。

As a reminder, the calculation of entanglement entropy schematically amounts to

提醒一下，纠缠熵的计算粗略来讲就是

$$G_R \to i\Delta \to W_{SJ} \to S \tag{26}$$

However, note that the scheme above does not tell one how to obtain the retarded Green function, G_R , which is the starting point. In fact, in general there exists no known expression for G_R in terms of quantities intrinsic to the causal set (such as the link or causal matrices). The examples below represent some important and interesting cases where an expression for G_R is known.

但需要注意，上述方案并没有说明如何得到作为起点的推迟格林函数 G_R 。事实上，一般来说目前还没有已知的、用因果集内禀量 (比如链路或因果矩阵) 表示的 G_R 表达式。下面的例子给出了一些 G_R 表达式已知的重要且有趣的情形。

Causal Diamond in 1 + 1D Flat Spacetime

1+1 维平直时空的因果菱形

The most studied and well-understood setting for entanglement entropy in a causal set is the causal diamond in 1+1D. This setting also benefits from numerous analytic results from the continuum being known and available for comparison.

因果集中纠缠熵研究最多、理解最透彻的场景就是 1+1 维的因果菱形。该场景还拥有众多连续统解析结果，可供对比参考。

We will mainly focus on the massless theory, for simplicity. The retarded Green function in 1 + 1D Minkowski spacetime, for a massless scalar field theory, is

为简单起见，我们将主要聚焦无质量理论。1 + 1D 闵氏时空中无质量标量场论的推迟格林函数为

$$G_R(x, x') = \frac{1}{2}\theta(t - t')\,\theta(\tau^2), \tag{27}$$

where $\tau = \sqrt{|t-t|^2 - |\mathbf{x} - \mathbf{x}'|^2}$ is the proper time between the two points and θ is the Heaviside step function. In other words, G_R is only nonzero if x' causally precedes x and when it does, G_R takes the value $\frac{1}{2}$. With G_R so directly related to the causal structure, we have an exact analogue of it in the causal set in terms of the causal matrix C . The causal matrix similarly has a nonzero value of 1 for each entry $C_{x'x}$ where x' causally precedes x or $x' \prec x$. Therefore, all we have to do is multiply the causal matrix by the constant $\frac{1}{2}$ to get the analogue of the retarded Green function in the causal set:

其中 $\tau = \sqrt{|t-t|^2 - |\mathbf{x}-\mathbf{x}'|^2}$ 是两点间的固有时，θ 是亥维赛阶跃函数。换句话说，仅当 x' 因果先于 x 时 G_R 非零，此时 G_R 的取值为 $\frac{1}{2}$ 。由于 G_R 与因果结构直接相关，我们在因果集中可以通过因果矩阵 C 得到它的精确对应。类似地，对每个满足 x' 因果先于 x 或 $x' \prec x$ 的条目 $C_{x'x}$ ，因果矩阵的非零取值为 1。因此，我们只需将因果矩阵乘以常数 $\frac{1}{2}$ ，就能得到因果集中推迟格林函数的对应形式:

$$G_R = \frac{1}{2}C \tag{28}$$

Next, we form $i\Delta$ from $\frac{i}{2}$ times the antisymmetric part of the causal matrix (When we write C_{xy} , we mean the entry corresponding to elements x and y in a point basis representation of the matrix C .):

接下来，我们由因果矩阵的反对称部分乘以 $\frac{i}{2}$ 构造出 $i\Delta$ (当我们写下 C_{xy} 时，指的是矩阵 C 点基表示中对应元素 x 和 y 的条目):

$$i\Delta_{xx'} = \frac{i}{2}\left(C_{x'x} - C_{xx'}\right). \tag{29}$$

From here it is a simple algebraic exercise to diagonalize (29) and restrict to its positive eigenspace to define W_{SJ} . The SJ state in the $1+1$D causal diamond has been extensively studied [17]. It resembles the standard Minkowski Wightman function (with an IR cutoff) away from the boundaries of the diamond. Near the left and right corners, it resembles the Minkowski Wightman function with a static mirror at these corners.

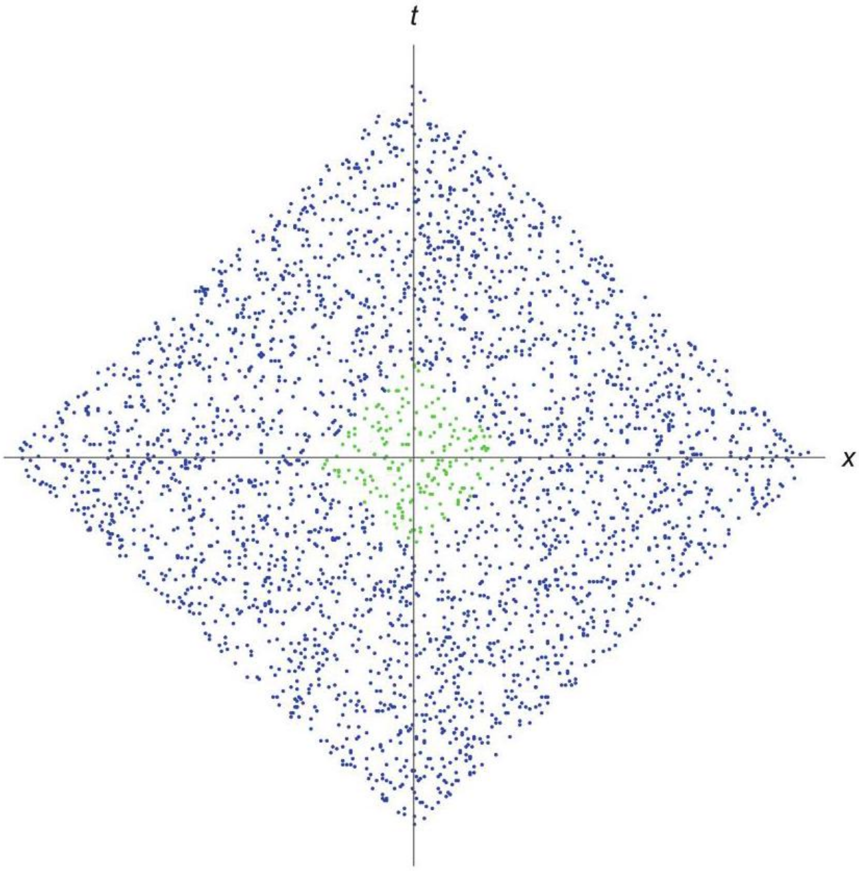
至此，对角化 (29) 并限制到其正本征空间来定义 W_{SJ} 只是一个简单的代数练习。$1+1$D 因果菱形中的 SJ 态已经得到了广泛研究 [17]。在远离菱形边界的区域，它类似带红外截断的标准闵氏威格曼函数；在左右角附近，它类似这些角处带有静态镜面的闵氏威格曼函数。

In [18, 19] it was shown that if we go ahead and compute the entanglement entropy according to the steps in section "Entropy from Correlation Functions," we obtain an unexpected answer: instead of the usual spatial area law scaling of the entanglement entropy with the UV cutoff, we obtain a spacetime volume scaling. Since we have $N \propto V$ in a causal set, a volume scaling means that the entanglement entropy scales linearly with N instead of the expected N^{D-2} scaling in $D > 2$ or logarithmic scaling when $D = 2$.

文献 [18, 19] 表明，如果我们按照“关联函数导出熵”一节中的步骤计算纠缠熵，会得到一个出乎意料的结果: 我们得到的是时空体积标度，而非纠缠熵随紫外截断的通常空间面积律标度。由于因果集中存在 $N \propto V$ ，体积标度意味着纠缠熵随 N 线性标度，而非预期中 $D > 2$ 下的 N^{D-2} 标度，也不是 $D = 2$ 时的对数标度。

More specifically, to study the scaling of the entanglement entropy with the UV cutoff, which in length dimensions is $\rho^{-1/D}$ (where ρ is now the sprinkling density), we fix the geometry into which the sprinkling is performed (e.g., the blue diamond in Fig. 4), as well as the entangling subregion (e.g., the green subdiamond in Fig. 4). We then vary the number of elements N that we sprinkle, thereby varying ρ and the UV cutoff. Figure 5 shows an example of the volume law scaling obtained, for the diamond setup shown in Fig. 4.

更具体地说，为了研究纠缠熵随紫外截断的标度行为 (紫外截断的长度量纲为 $\rho^{-1/D}$ ，其中 ρ 是撒播密度)，我们固定撒播的背景几何 (例如图 4 中的蓝色菱形) 以及纠缠子区域 (例如图 4 中的绿色子菱形)，随后改变撒播的元素数量 N ，以此改变 ρ 和紫外截断。图 5 给出了图 4 菱形设置下得到的体积律标度示例。



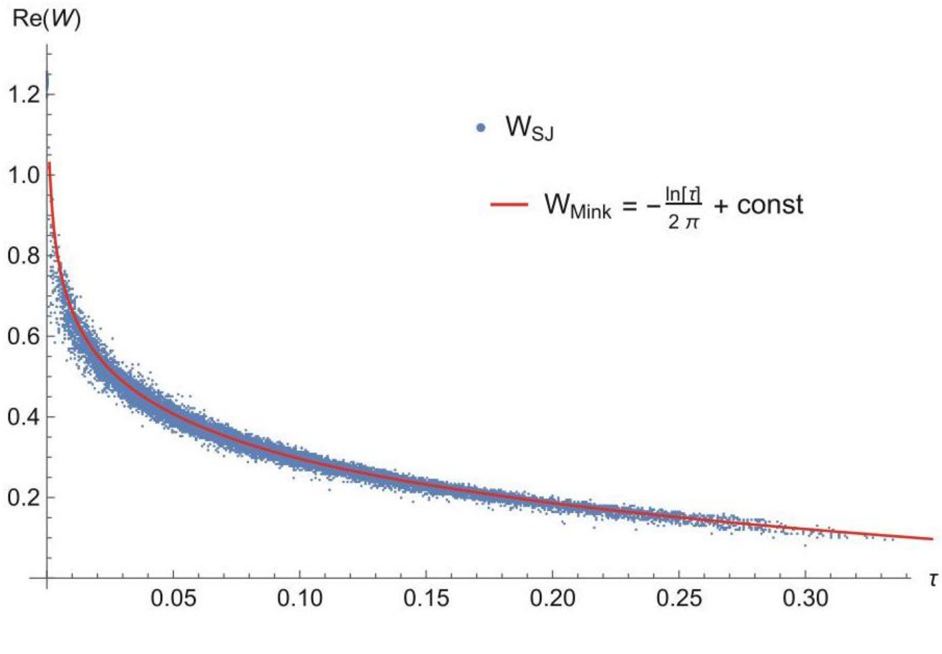


Fig. 4 The lower plot shows the SJ Wightman function in a causal set causal diamond, away from the boundaries. The SJ state is computed in the larger blue diamond at the top. The values of W_{SJ} versus proper time have been shown for elements in the inner Green subdiamond. Agreement is seen with the Wightman function associated with the IR-regulated Minkowski vacuum state (red curve)

图 4 下图展示了远离边界处因果集因果菱形中的 SJ 威格曼函数。SJ 态是在上方的蓝色大菱形中计算得到的，图中给出了内部绿色子菱形中元素的 W_{SJ} 随固有时的变化，结果与红外调节的闵氏真空态对应的威格曼函数 (红色曲线) 一致

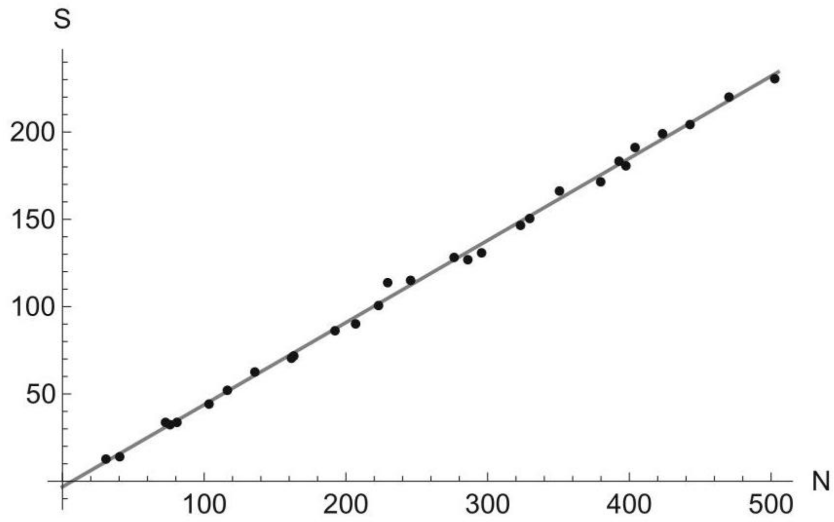


Fig. 5 The entanglement entropy grows linearly with N , demonstrating a spacetime volume law scaling. The points represent entropy values from the calculation of (4) in the setup of the upper plot in Fig. 4. The line is a linear fit to the data. In this example, the ratio of the side lengths of the two diamonds was $\frac{1}{4}$

图 5 纠缠熵随 N 线性增长，证明满足时空体积律标度。点代表图 4 上图设置中式 (4) 计算得到的熵值，线是对数据的线性拟合。本例中，两个菱形的边长比为 $\frac{1}{4}$

This result is peculiar to the causal set calculation, as the continuum analogue of the same calculation (performed in [20]) showed no sign of this behavior. A closer look at the eigenvalues of $i\Delta$, with the help of insight from analytic results from the continuum, reveals the source of the extra entropy. In the continuum diamond with side length 2ℓ , the eigenfunctions of $i\Delta$ with nonzero eigenvalues are [21]

该结果是因果集计算特有的，因为文献 [20] 中完成的同一计算的连续统版本未显示出该行为的迹象。借助连续统解析结果的启发，仔细观察 $i\Delta$ 的本征值，即可发现额外熵的来源。对于边长为 2ℓ 的连续统菱形，具有非零本征值的 $i\Delta$ 本征函数为 [21]

$$f_k(u,v) = e^{-iku} - e^{-ikv}, \text{ with } k = \frac{n\pi}{\ell}, n = \pm 1, \pm 2, \ldots \tag{30}$$

$$g_k(u,v) = e^{-iku} + e^{-ikv} - 2\cos(k\ell), \text{ with } k \mid \tan(k\ell) = 2k\ell \wedge k \in \mathbb{R} \tag{31}$$

The eigenfunctions above have been expressed in terms of lightcone coordinates $u = \frac{t+x}{\sqrt{2}}$ and $v = \frac{t-x}{\sqrt{2}}$. The eigenvalues are $\tilde{\lambda}_k = \ell/k$, with eigenvalues from both sets of eigenfunctions f_k and g_k approaching $\tilde{\lambda}_k = \frac{\ell^2}{n\pi}$ in the large k limit. The eigenfunctions (30) and (31) span the solutions of the Klein-Gordon equation ($\mathrm{Ker}\left(\Box + m^2\right) = \overline{\mathrm{Im}\,(\Delta)}$.). Therefore, if we were to consider a finite number of eigenfunctions up to some $k_{\max}$, we can think of $k_{\max}$ as a cutoff. In other words, with a finite set of eigenfunctions up to $k_{\max}$, we would only be able to expand solutions (e.g., initial data) up to this maximum wave number. Turning this argument around, if there were reason to believe that solutions beyond some UV cutoff $k_{\max}$ ought not to be supported in a setting, we would expect to obtain a finite number $\sim n_{\max}$ of eigenvalues and eigenfunctions corresponding to $k_{\max} = \frac{n_{\max}\pi}{\ell}$. This is precisely the scenario we are faced with in causal sets. In our 1 + 1D diamond causal set, the discreteness length is $\frac{1}{\sqrt{\rho}} = \sqrt{\frac{V}{N}} = \frac{2\ell}{\sqrt{N}}$. Hence, we do not expect to be able to meaningfully describe wavelengths shorter than this. Converting this wavelength to a wave number, we get $\frac{2\ell}{\sqrt{N}} = \frac{2\pi}{k_{\max}} \Rightarrow k_{\max} = \frac{\pi\sqrt{N}}{\ell} \Rightarrow n_{\max} \approx 2\sqrt{N}$, where the factor of 2 in $2\sqrt{N}$ comes from the fact that we have two sets of eigenfunctions f_k and g_k that will each have this $k_{\max}$ and contribute $\sqrt{N}$.

上述本征函数已用光锥坐标 $u = \frac{t+x}{\sqrt{2}}$ 和 $v = \frac{t-x}{\sqrt{2}}$ 表示。本征值为 $\tilde{\lambda}_k = \ell/k$，两组本征函数 f_k 和 g_k 的本征值在大 k 极限下都趋近于 $\tilde{\lambda}_k = \frac{\ell^2}{n\pi}$。本征函数 (30) 和 (31) 张成了克莱因-戈登方程 ($\mathrm{Ker}\left(\Box + m^2\right) = \overline{\mathrm{Im}\,(\Delta)}$.) 的解空间。因此，如果我们考虑到某一 $k_{\max}$ 为止的有限个本征函数，就可以将 $k_{\max}$ 视作截断。换言之，采用到 $k_{\max}$ 为止的有限本征函数集，我们只能展开到该最大波数为止的解 (例如初始数据)。反过来说，如果有理由认为超出某紫外截断 $k_{\max}$ 的解不应当出现在该场景中，我们就可以预期得到有限个 $\sim n_{\max}$ 对应于 $k_{\max} = \frac{n_{\max}\pi}{\ell}$ 的本征值和本征函数。这正是我们在因果集中遇到的情况。在我们的 1+1 维菱形因果集中，离散长度为 $\frac{1}{\sqrt{\rho}} = \sqrt{\frac{V}{N}} = \frac{2\ell}{\sqrt{N}}$，因此我们无法有效描述比这更短的波长。将该波长转换为波数，我们得到 $\frac{2\ell}{\sqrt{N}} = \frac{2\pi}{k_{\max}} \Rightarrow k_{\max} = \frac{\pi\sqrt{N}}{\ell} \Rightarrow n_{\max} \approx 2\sqrt{N}$，其中 $2\sqrt{N}$ 中的因子 2 来自我们有两组本征函数 f_k 和 g_k，每组都有这个 $k_{\max}$，总贡献为 $\sqrt{N}$。

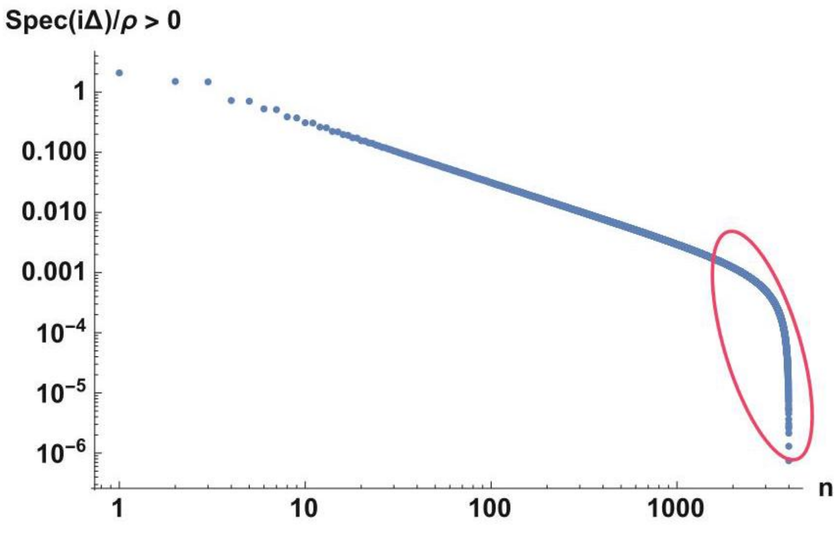


Fig. 6 The positive eigenvalues of $i\Delta$ for a sample sprinkling into a causal diamond.

图 6 随机撒入因果菱形的样本的 $i\Delta$ 的正本征值

However, a look at the eigenvalues and eigenfunctions of $i\Delta$ in the causal set reveals that we in fact end up with many more nonzero eigenvalues than $n_{\max}$. An example of the spectrum of $i\Delta$ is shown in Fig. 6. The values of the positive eigenvalues are shown on a log-log scale, where they have been ordered from largest to smallest, and each i th eigenvalue in this ordering is paired with i on the horizontal axis. Circled are the extra eigenvalues that are beyond the expected $n_{\max}$. Interestingly, these extra eigenvalues behave qualitatively differently from the rest: they do not follow a power law like the larger eigenvalues.

然而，观察因果集中 $i\Delta$ 的本征值和本征函数可以发现，我们实际上得到的非零本征值比 $n_{\max}$ 多得多。$i\Delta$ 谱的一个实例如图 6 所示。正本征值的值绘制在双对数坐标上，本征值按从大到小排序，该排序中每第 i 个本征值对应横轴的 i 。圈出的是超出预期数量 $n_{\max}$ 的额外本征值。有意思的是，这些额外本征值的定性行为和其他本征值不同: 它们不遵循大本征值满足的幂律。

When the contributions to the entanglement entropy from these extra eigenvalues and their corresponding eigenvectors are removed, we recover the expected spatial area law scaling of the entanglement entropy with respect to the UV cutoff. This removal is referred to as a truncation, and it must be implemented at two stages of the calculation: (1) a first truncation when $W_{SJ} = \mathrm{Pos}\,(i\Delta)$ is constructed and (2) a second truncation when the generalized eigenvalue equation (5) is solved. These truncations can also be regarded as projections down to the subspace where $k_{\max} = \frac{\pi\sqrt{N_\diamond}}{\ell_\diamond}$, where the diamond subscript indicates that in the first truncation which is in the global diamond, N and ℓ are the total number of elements and size of this diamond, whereas in the second truncation which occurs in the smaller subdiamond, N and ℓ are the number of elements in and size of the subdiamond. When this so-called double truncationis performed, we obtain the result shown in Fig. 7

移除这些额外本征值及其对应本征向量对纠缠熵的贡献后, 我们就得到了纠缠熵关于紫外截断的预期空间面积律标度。这种移除称为截断，必须在计算的两个阶段执行:(1) 构造 $W_{SJ} = \mathrm{Pos}\,(i\Delta)$ 时的第一次截断，(2) 求解广义本征值方程 (5) 时的第二次截断。这些截断也可视为向 $k_{\max} = \frac{\pi\sqrt{N_\diamond}}{\ell_\diamond}$ 所在子空间的投影，其中菱形下标表示: 第一次截断在全局菱形内进行，N 和 ℓ 分别是该菱形的总元数和尺寸；而第二次截断在更小的子菱形内进行，N 和 ℓ 分别是子菱形的元数和尺寸。完成这种所谓的双截断后，我们得到了图 7 所示的结果

Note that, as mentioned earlier, a logarithmic scaling with respect to the UV cutoff is the expected ”area law” result in $1+1$D [22], and this is what is obtained after the double truncation. A coefficient of $\frac{1}{3}$ to the logarithmic scaling is also an expected (A coefficient of $\frac{1}{3}$ is expected in the case of two spatial boundaries (such as in the configuration of Fig.4). The case of one spatial boundary, where a coefficient of $\frac{1}{6}$ is expected, was also studied in [23] and agreement with a logarithmic scaling and the $\frac{1}{6}$ coefficient was confirmed.) universal constant [24] that the causal set results agree with.

请注意，如前文所述，在 $1+1$D 中，关于紫外截断的对数标度就是预期的“面积律”结果 [22]，这正是双截断后得到的结论。对数标度的 $\frac{1}{3}$ 系数也是一个预期的 (存在两个空间边界时 (例如图 4 的构型) 预期系数为 $\frac{1}{3}$ 。存在一个空间边界的情况预期系数为 $\frac{1}{6}$ ，该情况也已在文献 [23] 中研究，证实了结果符合对数标度和 $\frac{1}{6}$ 系数。) 普适常数 [24]，因果集结果与该常数一致。

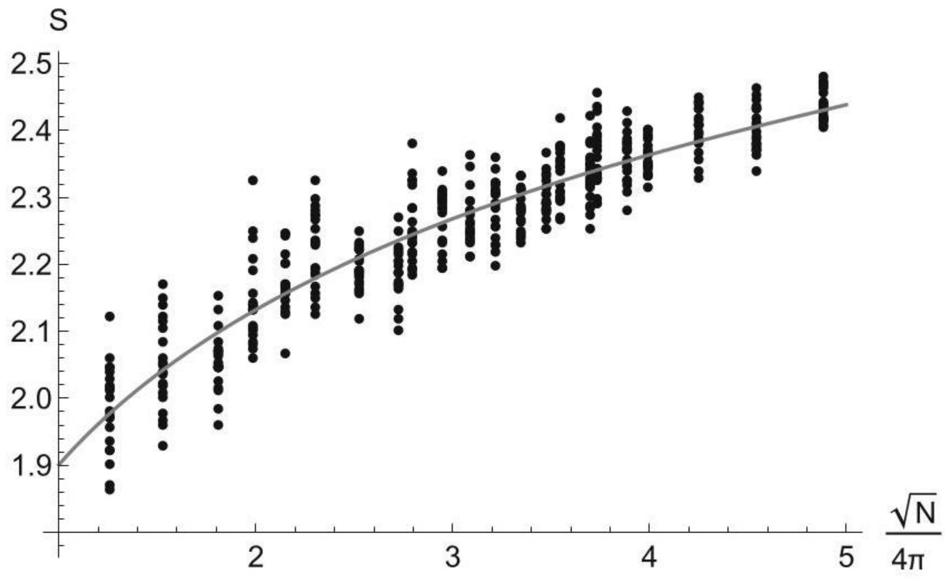


Fig. 7 The entanglement entropy versus the UV cutoff, following a logarithmic scaling with a coefficient consistent with $\frac{1}{3}$. This is the conventional result according to the expectation of a spatial area law

图 7 纠缠熵随紫外截断变化，遵循对数标度，其系数与 $\frac{1}{3}$ 一致，符合空间面积律预期的常规结果

These results in the causal diamond were extended to the massive scalar field theory in [25]. In the massive theory, because mass is an intermediate scale lying between the UV (discreteness scale) and IR (diamond size) scales, the same double truncation procedure of the massless theory can be used. This is extremely useful as we lack analytic results in the massive theory to otherwise give us some guidance toward the nature of the eigenvalues. In [25] scalings of the entanglement entropy with both the UV cutoff and mass were studied, and in both cases the expected result of logarithmic scaling with a coefficient of $\frac{1}{3}$ was obtained. In the same work, the entanglement entropy results were also extended to Rényi entropies [26]. The Rényi entropy of order q , in terms of the quantities we are working with in our formulation, is given by [18, 25]

本文 [25] 将因果菱形中的这些结果推广到了有质量标量场论。在有质量理论中，由于质量是介于紫外 (离散标度) 和红外 (菱形尺寸) 之间的中间标度，因此可以沿用无质量理论的相同双截断步骤。由于有质量理论缺少解析结果来为我们指明本征值的性质，因此该方法极为有用。文献 [25] 研究了纠缠熵分别随紫外截断和质量的标度行为，两种情况都得到了预期结果: 系数为 $\frac{1}{3}$ 的对数标度。同一工作还将纠缠熵结果推广到了雷尼熵 [26]。在我们公式所用的物理量框架下，q 阶雷尼熵可表示为 [18, 25]

$$S^{(q)} = \frac{-1}{1-q} \sum_{\lambda} \ln\left(\lambda^q - (\lambda - 1)^q\right). \tag{32}$$

The solutions to (5) come in pairs of λ and $1 - \lambda$ and $|\lambda| \geq 1$. Each term in the sum (32) represents the contribution from one such pair. Similarly, other measures of entropy such as Tsallis entropy [27] can also be calculated and studied in this manner.

方程 (5) 的解以 λ、$1-\lambda$ 和 $|\lambda| \geq 1$ 对的形式出现，求和式 (32) 中的每一项都对应一对解的贡献。同理，其他熵测度例如 Tsallis 熵 [27] 也可以用这种方法计算和研究。

Disjoint Causal Diamond Regions

不相交因果菱形区域

Another setting in which entanglement entropy in causal set theory has been studied is that of disjoint causal diamonds within a larger global causal diamond in $1+1$D [23]. An example setup with two disjoint subdiamonds is shown in Fig. 8. Entanglement entropy of disjoint regions has been studied in several places in the literature [28-30], and the calculations tend to be quite involved and difficult. In contrast, the calculation using (5) and (4) for the disjoint diamonds is very similar to the calculation for the single diamond, which is now well-understood and relatively easy to do. Therefore, this is an example where there are calculational advantages, in addition to physical ones, to working with the spacetime formulation of section "Entropy from Correlation Functions" in a causal set.

因果集理论中研究纠缠熵的另一框架是 $1+1$D 中更大全局因果菱形内的不相交因果菱形 [23]。图 8 展示了包含两个不相交子菱形的示例设置。不相交区域的纠缠熵已在多篇文献 [28-30] 中得到研究，这类计算通常相当复杂困难。相比之下，利用式 (5) 和式 (4) 对不相交菱形做的计算与单菱形的计算非常相似，而单菱形计算目前已经得到充分理解，操作相对简单。因此，在因果集内使用“关联函数导出熵”一节的时空公式，除物理优势外还具备计算优势，本例就是很好的说明。

In [23] explicit calculations were done for the case of two and three disjoint subdiamonds; the entanglement entropy scalings in both of these cases were shown to be consistent with the logarithmic scalings expected. While scalings with respect to the UV cutoff were the focus of [23], there are several other scales in the problem (e.g., the sizes of the subdiamonds, the separation(s) between the diamonds, the distance away from the boundary of the global diamond, etc.) whose relation to the entanglement entropy would be interesting to investigate.

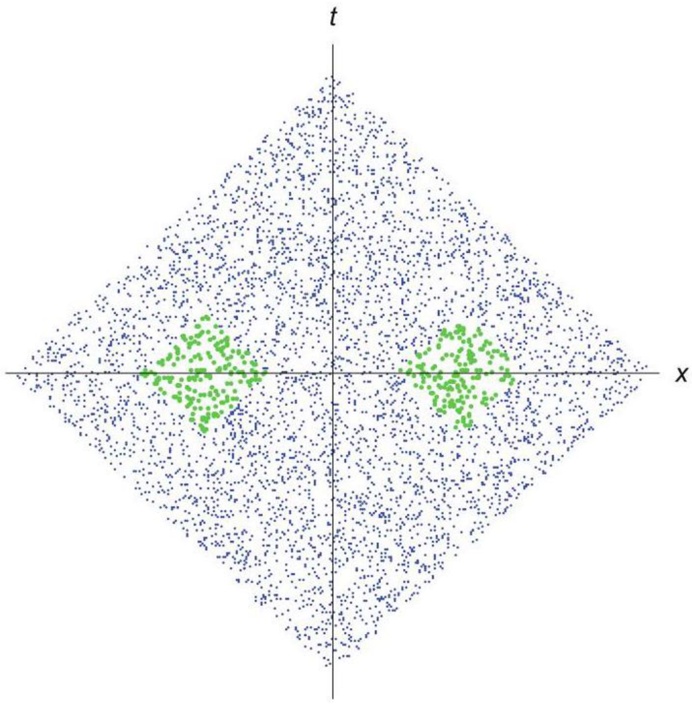
文献 [23] 对两个和三个不相交子菱形的情况做了显式计算；结果表明这两种情况下的纠缠熵标度都和预期的对数标度一致。尽管文献 [23] 的研究重点是纠缠熵相对于紫外截断的标度，但该问题还存在多个其他特征标度 (例如子菱形的尺寸、菱形之间的间隔、菱形到全局菱形边界的距离等)，研究这些标度和纠缠熵的关系会很有意义。

The mutual information for a two-subdiamond setup was also studied in [23]. The mutual information in this case is the difference between the entanglement entropy associated with the union of the two subdiamonds and the sum of the entropies of the individual subdiamonds. Specifically, the relation between the mutual information and the separation distance between two diamonds was studied. The results demonstrated the expected qualitative behavior that the mutual information asymptotically vanishes as the separation between the subdiamonds grows and diverges as the separation goes to zero.

文献 [23] 还研究了双子菱形设置的互信息。该情况下的互信息是两个子菱形并集的纠缠熵，减去两个子菱形各自纠缠熵的和。文献具体研究了互信息与两菱形间隔距离的关系，结果展现出预期的定性行为: 互信息随子菱形间隔增大渐近趋于零，随间隔趋近于零发散。

Fig. 8 Two smaller causal diamonds within a larger causal diamond

图 8 大因果菱形内部的两个较小因果菱形



De Sitter Spacetime

德西特时空

Cosmological event horizons, just like black hole event horizons, also have a classical entropy, the Gibbons-Hawking entropy [31], associated with them that scales as their spatial area. Hence, applications of entanglement entropy to cosmological spacetimes are of particular interest. De Sitter spacetime offers an especially convenient setting to study the entanglement entropy. This is partly due to its maximal symmetry, which makes sprinkling into it considerably easier in comparison with sprinkling into more generic curved spacetimes. Of course, any sprinkling into de Sitter spacetime would not represent the full global de Sitter spacetime, as that has infinite volume and would therefore require an infinite number of elements, which is computationally not feasible. Instead, sprinklings into finite slabs of global de Sitter spacetime are used, and it is ensured that any results obtained are stable under making the volume of the slab larger and larger.

宇宙学事件视界与黑洞事件视界类似，也具有与自身关联的经典熵，即吉本斯-霍金熵 [31]，该熵随视界的空间面积缩放。因此，将纠缠熵应用于宇宙学时空中具有特殊的研究意义。德西特时空为研究纠缠熵提供了一个格外便利的框架。这部分得益于它的最大对称性，相较于向更一般的弯曲时空撒播，向德西特时空撒播要简单得多。当然，向德西特时空撒播无法覆盖完整的整体德西特时空，因为整体德西特时空体积无限，因此需要无限多个元素，这在计算上不可行。实际研究中采用的是向整体德西特时空的有限切片撒播，且已证实在切片体积不断增大的情况下，所得结果始终保持稳定。

Another attractive feature of working with de Sitter spacetime is that an expression for the retarded Green function, in terms of causal set quantities, is known in this context [32]. This gives us the starting point in (26). The SJ Wightman function in causal sets sprinkled into de Sitter spacetime was studied in [33]. The entanglement entropy in causal set sprinklings of de Sitter slabs, using the formalism reviewed in this chapter, was studied in [34]. In particular the entanglement entropy associated with the subregion within the horizon of a static observer at the North Pole was considered. This subregion and its causal complement are shown in the conformal diagram in Fig. 9.

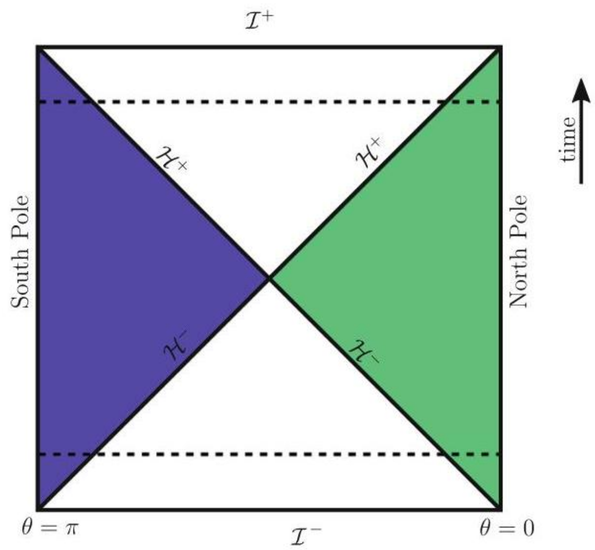
研究德西特时空的另一个优势在于，该背景下以因果集合量表示的推迟格林函数表达式是已知的 [32]，这为我们给出了式 (26) 的出发点。文献 [33] 研究了撒入德西特时空的因果集合中的 SJ 威曼函数。文献 [34] 利用本章回顾的形式体系，研究了德西特切片因果集合撒播中的纠缠熵，具体研究了北极处静态观者视界内子区域的纠缠熵。该子区域及其因果补集见图 9 的共形图。

Similar to the case of the causal diamond, the spectrum of $i\Delta$ has two characteristic regimes: a branch of eigenvalues that are larger in magnitude and follow a power law and a branch of more numerous small but nonzero eigenvalues that do not follow a power law. Without truncating away this second branch, once again a spacetime volume scaling is obtained. A spatial area scaling is recovered only after implementing a double truncation. However, choosing a precise double truncation scheme is more subtle in this case, as we lack analytic results in de Sitter spacetime to guide us. In other words, we do not know exactly how the eigenvalues of $i\Delta$ relate to something like a $k_{\max}$ in de Sitter space. In the absence of analytic results to guide us, one can estimate the transition between the power law regime and nonpower law regime using a number of different strategies. Some of these strategies were studied in [34] and shown to yield spatial area laws. However, in that work it was found challenging to hone in on a unique prescription for the truncations, as many different choices yielded area laws. On the other hand, complementarity of the entanglement entropy was found to be a property that was difficult to preserve, since in the second truncation it was unclear which (nonunique) truncation in a subregion ought to be paired with which (nonunique) counterpart truncation in the causally complementary subregion. We will return to this point of finding a general truncation scheme in the concluding section of this chapter.

与因果菱形的情况类似, $i\Delta$ 的谱存在两个特征区域: 一个分支的本征值幅值更大, 满足幂律分布; 另一个分支包含数量更多的小非零本征值，不满足幂律分布。若不截去第二个分支，会再次得到时空体积标度。只有实施双重截断后，才能得到空间面积标度。但在德西特情况下，确定精确的双重截断方案更为棘手，因为我们缺乏德西特时空的解析结果作为指导。换言之，我们并不清楚 $i\Delta$ 的本征值具体如何对应德西特空间中类似 $k_{\max}$ 的物理量。在没有解析结果指导的情况下，可以通过多种不同策略估计幂律区域和非幂律区域之间的转变点。文献 [34] 研究了部分此类策略，证明这些策略可以得到空间面积律。但该研究发现，很难确定唯一的截断方案，因为多种不同选择都能得到面积律。另一方面，纠缠熵的互补性是一个难以保留的性质，因为在第二次截断中，并不清楚子区域中的哪个 (非唯一) 截断应当与因果补子区域中的哪个 (非唯一) 对应截断配对。我们会在本章的结论部分回到寻找通用截断方案这一问题上。

Fig. 9 Entangled wedges corresponding to the horizon of a static observer at the North Pole (green) and its causal complement (purple). The dashed lines represent the boundaries of the slab, $-T_{\max} < T < T_{\max}$, in de Sitter spacetime $ds^2 = \frac{1}{\cos^2 T}\left(-dT^2 + d\Omega^2_{d-1}\right)$

图 9 对应北极处静态观者视界的纠缠楔 (绿色) 及其因果补集 (紫色)。虚线代表德西特时空 $ds^2 = \frac{1}{\cos^2 T}\left(-dT^2 + d\Omega^2_{d-1}\right)$ 中切片 $-T_{\max} < T < T_{\max}$ 的边界



Nonlocal Quantum Field Theory

非局域量子场论

Ordinarily, the retarded Green function which is the starting point of (26) would be obtained via the Klein-Gordon equation in the continuum. In causal set theory, we do not have a local field equation that is the analogue of the Klein-Gordon equation. Instead, we must obtain G_R through other means. There is no general recipe in causal set theory for deriving G_R . Expressions for it are known in a few distinct cases with help from the continuum analogues of these Green function and dimensional analysis.

通常而言，式 (26) 的起点推迟格林函数可通过连续统中的克莱因-戈登方程得到。在因果集合论中，我们不存在对应克莱因-戈登方程的局域场方程，因此必须通过其他方式得到 G_R 。因果集合论中尚无推导 G_R 的通用方法，仅在少数特殊情形中，借助这些格林函数的连续统类比和量纲分析，才得到了它的表达式。

While a local analogue of the d'Alembertian $\Box$ and therefore the Klein-Gordon equation does not exist in causal set theory, a nonlocal analogue of it does [35-38]. In fact a whole family $\Box_k$ of nonlocal d' Alembertians exists, with each member distinguished by a nonlocality scale ℓ_k . For example, in $1+1$D, $\Box_k \phi$ at the element $x \in \mathcal{C}$ is defined to be [35]

尽管因果集合论中不存在达朗贝尔算符 $\Box$ (以及克莱因-戈登方程) 的局域类比，但确实存在它的非局域类比 [35-38]。实际上，因果集合论中存在一整个非局域达朗贝尔算符族 $\Box_k$ ，每个成员由一个非定域性标度 ℓ_k 区分。例如，$1+1$D, $\Box_k \phi$ 中元素 $x \in \mathcal{C}$ 处的达朗贝尔算符定义为 [35]

$$\Box_k \phi(x) = \frac{4\varepsilon}{\ell_\rho^2} \left(\frac{1}{2} \phi(x) + \varepsilon \sum_{y \prec x} f(n(x,y), \varepsilon) \phi(y) \right), \tag{33}$$

where ℓ_ρ is the discreteness scale, $\varepsilon \equiv \ell_\rho^2 \ell_k, n(x,y)$ is the number of elements in the causal diamond between x and y , and

其中 ℓ_ρ 是离散标度，$\varepsilon \equiv \ell_\rho^2 \ell_k, n(x,y)$ 是 x 与 y 之间因果菱形中的元素数量，且

$$f(n, \varepsilon) = (1-\varepsilon)^n \left(1 - \frac{2\varepsilon n}{1-\varepsilon} + \frac{\varepsilon^2 n(n-1)}{2(1-\varepsilon)^2} \right). \tag{34}$$

The nonlocality of (33) is evident in the fact that in order to know the action of the d' Alembertian on the field at the point x , we must consider a sum of quantities involving a set of other elements y to the past of x . Therefore, this nonlocality has a causal, or more specifically retarded, nature.

式 (33) 的非局域性十分明显: 要得知达朗贝尔算符作用在点 x 处场的结果，我们必须对包含 x 过去方向一系列其他元素 y 的量求和。因此，这种非局域性具有因果性，更准确地说，具有推迟性。

In the infinite density limit $(\ell_\rho \to 0)$, the mean of $\Box_k$ over all sprinklings into a spacetime reduces to the usual continuum d'Alembertian plus a term containing the Ricci scalar curvature [36]:

在无穷密度极限 $(\ell_\rho \to 0)$ 下，对所有洒入时空的过程求平均后，$\Box_k$ 退化为常规连续统达朗贝尔算符加上一个包含里奇标曲率的项 [36]:

$$\lim_{\ell_\rho \to 0} \overline{\Box}_k \phi(x) = \left(\Box - \frac{1}{2} R(x) \right) \phi(x). \tag{35}$$

With these nonlocal retarded causal set d'Alembertians at hand, we can now invert them to obtain their corresponding retarded Green functions $G_{R,k}$, for use in (26). This was done in [39] for nested causal diamonds in $1+1, 2+1$, and $3+1$-dimensional Minkowski spacetime. Once again, as in the local calculations, only after the use of a double truncation, the expected spatial area scalings were obtained.

得到这些非局域推迟因果集合达朗贝尔算符后，我们现在可以对其求逆，得到对应的推迟格林函数 $G_{R,k}$ ，以供式 (26) 使用。文献 [39] 已针对 $1+1, 2+1$ 中的嵌套因果菱形和 3+1 维闵氏时空完成了这一工作。与局域计算的情形一样，只有经过双截断处理后，才能得到预期的空间面积标度。

Discussion and Outlook

讨论与展望

Entanglement entropy in causal set theory is a powerful way to covariantly and unambiguously count the quantum field degrees of freedom one has access to in settings such as spacetimes with event horizons. One must know the scalar field retarded Green function in the spacetime of interest in order to carry out the entanglement entropy calculations, as well as a double truncation rule in order to project out the irrelevant degrees of freedom in the causal set.

因果集理论中的纠缠熵是一种强有力的方法，能够协变且明确地计数在带有事件视界的时空这类场景中可获取的量子场自由度。要进行纠缠熵计算，必须得到目标时空中标量场的推迟格林函数，还需要一条双截断规则来投影出因果集中无关的自由度。

An expression for the retarded Green function in causal sets approximated by generic curved spacetimes, in terms of quantities intrinsic to the causal set, is not at present known. Such an expression is known in a few cases, such as the Minkowski and de Sitter examples reviewed above. Nonlocal versions of these Green functions, as discussed in section "Nonlocal Quantum Field Theory," can be computed more generally. Alternatively, viewing the causal set as a Lorentzian and covariant discretization of the continuum, we can also simply take the Green functions and/or correlator expressions from the continuum and restrict them to the causal set elements. Thereby we would be regulating any coincidence limit divergences that may be present, by imposing the minimum distance set by the causal set discreteness scale.

目前，针对由一般弯曲时空近似得到的因果集，尚未得到完全由因果集内禀量表示的推迟格林函数表达式。仅在少数情形下存在这样的表达式，比如上文回顾的闵氏时空和德西特时空例子。正如“非定域量子场论”一节所讨论的，这些格林函数的非定域形式可以在更一般的情况下计算得到。或者，将因果集视为连续统的洛伦兹协变离散化，我们也可以直接从连续统中获取格林函数和/或关联函数表达式，再将其限制在因果集元素上。通过这种方式，我们可以借助因果集离散度规设定的最小距离，规整化可能存在的任何重合极限发散。

As mentioned, it is also necessary to have a prescription for the double truncation in order to meaningfully study entanglement entropy in causal set theory. This prescription is well-understood in the massless theory in causal diamonds in $1+1$D flat spacetime. As shown in [23] and [25], the same prescription can be used for the case of multiple disjoint causal diamonds as well as the massive scalar field theory. More generally, the same prescription can be used in any $1+1$D scalar field theory (e.g., the nonlocal theory or in curved spacetimes), as long as the intermediate scale is far from the discreteness scale. This is because the truncations concern the deep UV regime of the theory, which has the same character in all theories where the other length scales are far from the discreteness scale.

如前所述，要在因果集理论中开展有意义的纠缠熵研究，还需要一套双截断方案。该方案在 $1+1$D 平直时空的因果钻石中的无质量理论里已经得到充分理解。正如文献 [23] 和 [25] 所示，这套相同的方案也可用于多个不相交因果钻石以及有质量标量场理论的情形。更一般地，只要中间尺度远大于离散尺度，这套相同的方案就可以用于任意 1+1 维标量场理论 (例如非定域理论或弯曲时空下的理论)。这是因为截断涉及理论的深紫外区域，当所有其他长度尺度都远离离散尺度时，该区域在所有理论中的性质都是一致的。

In [34] some generalizations of the $1+1$D causal diamond double truncation scheme were studied and applied to calculations in causal diamonds in $3+1$D Minkowski spacetime as well as slabs in de Sitter spacetime. One strategy was to keep $\alpha N^{\frac{D-1}{D}}$ of the largest eigenvalues and their corresponding eigenfunctions, where α is a constant that is a free parameter (several choices for α were investigated). This counting is motivated by the expectation that the number of independent degrees of freedom is the number that would lie within some approximate Cauchy surface-like submanifold (e.g., a thickened antichain). The number of elements in such a submanfiold would be proportional to its volume, which is approximately $V^{\frac{D-1}{D}}$. This strategy succeeds in yielding area laws, but it does not produce a unique prescription (many choices of α are possible), and complementarity is difficult to achieve. Another strategy was to try to estimate the transition between the power law to non-power law regime in the spectrum, through estimating when the approximate linear trend in Fig. 6 ends and begins to curve. This strategy sometimes succeeds in producing an area law, but it too suffers from an ambiguity in how and how sensitively to define the transition from power law (line on the log-log scale) to nonpower law (curve on the log-log scale). There are some other possible truncation schemes that would merit future investigation. For example, one ansatz could be that in the power law regime, each n th (positive) eigenvalue $\tilde{\lambda}_n$ of $i\Delta$ (in any dimension), when sorted from largest to smallest, is proportional to $\frac{1}{n^p}$, where the proportionality constant is given unambiguously by the value of the largest eigenvalue (where $n=1$) and p can be approximated from the spectrum. For example, we know that $p=1$ in the causal diamond in $1+1$D and $p=\frac{1}{2}$ in the causal diamond in $2+1$D . We can then choose $n_{\max}$ to be $N^{\frac{D-1}{D}}$ and estimate the magnitude of the smallest eigenvalue in the power law regime to then be $\tilde{\lambda}_{\min}=\frac{\tilde{\lambda}_{\max}}{N^{\frac{p(D-1)}{D}}}$.

文献 [34] 研究了 $1+1$D 因果菱形双重截断方案的若干推广，并将其应用于 $3+1$D 闵氏空间中因果菱形以及德西特空间中 slab 的计算。一种策略是保留最大特征值及其对应本征函数中的前 $\alpha N^{\frac{D-1}{D}}$ 个，其中 α 是常数，也是一个自由参数 (论文研究了 α 的多种取值)。这种计数方式的动机来自于: 独立自由度的数量应当是落在近似柯西面类子流形 (例如加粗的反链) 内的自由度数量。这类子流形的元素数量与其体积成正比，体积近似为 $V^{\frac{D-1}{D}}$ 。该策略可以成功得到面积定律，但无法给出唯一的方案 (α 存在多种可能取值)，也难以实现互补性。另一种策略是通过估计图 6 中近似线性趋势何时结束并开始弯曲，来估计谱中幂律区和非幂律区的转变。该策略有时能成功得到面积定律，但它同样存在歧义: 如何定义幂律 (双对数坐标上的直线) 到非幂律 (双对数坐标上的曲线) 的转变，以及该定义的敏感度如何，都没有确定结论。还有其他一些可能的截断方案值得未来研究。例如，可以提出这样一个假设: 在幂律区，将所有 (正) 特征值按从大到小排序后，任意维度下 $i\Delta$ 的每第 n 个特征值 $\tilde{\lambda}_n$ 都与 $\frac{1}{n^p}$ 成正比，比例常数由最大特征值 (满足 $n=1$) 唯一确定，且 p 可从谱中近似得到。例如，我们已知 $1+1$D 因果菱形中满足 $p=1$ ，$2+1$D 因果菱形中满足 $p=\frac{1}{2}$ 。我们之后可以取 $n_{\max}$ 为 $N^{\frac{D-1}{D}}$ ，估计出幂律区最小特征值的大小为 $\tilde{\lambda}_{\min}=\frac{\tilde{\lambda}_{\max}}{N^{\frac{p(D-1)}{D}}}$ 。

More analytic results for the eigenvalues of $i\Delta$ in the continuum would also aid our understanding of the eigenvalues in the causal set and better inform our truncation schemes. It is, however, quite difficult to analytically solve for the eigenfunctions of integral operators.

获得连续谱下 $i\Delta$ 特征值更多解析结果，也有助于我们理解因果集的特征值，并为我们的截断方案提供更多参考。但想要解析求解积分算子的本征函数是十分困难的。

Another perspective is that there should be no truncations and that all nonzero eigenvalues and eigenfunctions must contribute to the entropy [40]. However, even in this case, we must face the question of how small of an eigenvalue we can really expect to exist in the causal set calculations. Remember that we have the condition (6) that $i\Delta v \neq 0$. Due to the numerical nature of the calculations, there is always some degree of numerical error, and we must identify a threshold beyond which to set the values to zero. While doing so, we must also assess whether we are throwing away anything physical due to the numerical error. Therefore, a better understanding of the truncated contributions and what solutions can be meaningfully supported on the causal set is needed.

另一种观点认为不需要任何截断，所有非零特征值与本征函数都应当对熵有贡献 [40]。但即便在这种情况下，我们仍要面对一个问题: 在因果集计算中，我们实际能预期存在的特征值最小能到多少? 别忘了我们有条件 (6): $i\Delta v \neq 0$ 。由于计算是数值计算，总会存在一定程度的数值误差，我们必须划定一个阈值，将低于该阈值的值置零。在这个过程中，我们还必须评估我们是否会因数值误差丢弃掉任何物理内容。因此，我们需要更好地理解截断掉的贡献，以及哪些解是可以被因果集有意义地支撑的。

In [25] some insight was gained into the nature of the truncated contributions. Motivated by the observation that the untruncated contributions had continuum-like analogues whereas the truncated ones did not, as well as the observation that the truncated eigenfunctions had many sharp variations at the discreteness scale, it was investigated whether the truncated contributions may be fluctuations particular to a given sprinkling. Namely, it was investigated whether these contributions were random fluctuations that behaved differently from one sprinkling to the next, or whether they had features which persisted over an ensemble of different sprinklings. There are different prescriptions one can use to investigate this; one particular scheme, involving fixing a coarser sub-causal set in order to use it to take averages, was used in [25]. Indeed, evidence was found in favor of this conjecture that the truncated contributions are fluctuation-like, and the scheme studied in [25] indicated a transition point to the fluctuation-like regime that was consistent with the double truncation in the causal diamond in $1+1$D . This is promising both as insight into the nature of the truncated contributions and as practically in order to use it to inform a double truncation scheme in more general settings. The transition to fluctuation-like behavior can thus potentially be used in general to distinguish between contributions we must keep and ones we must not.

文献 [25] 对截断贡献的性质获得了一些洞察。基于以下两个观察: 未截断贡献存在连续体类似物而截断贡献没有，且截断本征函数在离散尺度存在许多剧烈变化，文献 [25] 研究了截断贡献是否可能是特定撒播所特有的涨落。具体而言，研究了这些贡献是在不同撒播中表现各异的随机涨落，还是存在跨不同撒播系综保持不变的特征。可以采用不同方案研究该问题，文献 [25] 使用了一种特定方案: 固定一个更粗的子因果集来进行平均。研究确实找到了支持截断贡献具有涨落性质这一猜想的证据，且文献 [25] 研究的方案指示出涨落区域存在一个转变点，该转变点与 $1+1$D 中因果菱形的双重截断一致。这无论是对理解截断贡献的性质，还是对后续将其用于指导更通用场景下的双重截断方案都很有前景。因此，向涨落行为的转变通常可潜在用于区分我们需要保留和需要舍弃的贡献。

There are many other applications of the entanglement entropy formulation reviewed in this chapter that would be interesting to explore in causal set theory. For example, up to first order in perturbation theory, the entanglement entropy for interacting scalar field theories such as those introduced in [41-43], can be studied. There is also an analogue of (5) for Fermionic field theories, except with the anti-commutator appearing instead of the commutator. Currently, there is no known construction of a Fermionic field theory on a causal set, in terms of quantities intrinsic to the causal set. When such a construction is available, the entanglement entropies of Fermionic field theories could also be studied.

本章综述的纠缠熵表述还有许多其他应用，值得在因果集理论中进一步探索。例如，在微扰论一阶范围内，可以研究文献 [41-43] 中引入的这类相互作用标量场论的纠缠熵。费米子场论也存在式 (5) 的类似物，只不过其中出现的是反对易子而非对易子。目前，在因果集框架内，尚未有从因果集内禀量出发构建费米子场论的已知方案。等到该构建方案完成后，就可以进一步研究费米子场论的纠缠熵了。

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